Technical Notes

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Mathematical Characteristics of Flows with Nonlinear and Nonequilibrium Transport

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Nomenclature

c = generalized scalar
Pe = $Pr \times Re$
Pr = ratio of diffusivities
q = flux of c
Re = Reynolds number
u = axial velocity
x = axial coordinate
y = normal coordinate
δ = sensitivity coefficient in active transport model, Eq. (10)
ϵ = relaxation constant
θ = function defined by Eq. (15)
λ = characteristics, $d\Psi/dx$
ν = kinematic viscosity
ρ = density
σ = $Pr-1$
τ = shear stress
Φ = function defined by Eq. (16)
Ψ = stream function

Subscripts

x,Ψ = differentiation with respect to x,Ψ
τ,q = momentum and scalar transport, respectively

Superscripts

$(\ )'$ = dimensional quantity
$(\tilde{\ })$ = nondimensional quantity
$0,1$ = zeroth- and first-order terms, respectively, in a series expansion

Formulation

NONLINEAR effects are of general importance regarding active scalar transport involving temperature- and species-dependent transport coefficients. Currently, interest is being revived in the use of additives injected into a turbulent boundary layer which impede turbulence production and, therefore, their own transport. In addition, turbulence modeling requires representation of the nonequilibrium (viscoelastic) nature of turbulent transport of entities which are, themselves, the agents of transport.[1-3] The mathematical and physical implications of such representations are therefore of widespread importance, and the development in Ref. 4 is extended to include nonlinear and nonequilibrium effects.

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The model problem studied herein is a laminar flow (or a laminarized view of turbulence with effective transport coefficients) sufficiently complex to expose the features of interest. Consider the evolution of a constant pressure thin shear flow. As shown in Ref. 4, the dimensional equations of momentum and stress (indicated by primed variables) may be written in streamfunction (Ψ,x) independent variables as:

$$\rho u_x' - \tau_\Psi' = 0 \tag{1}$$

$$\epsilon_\tau' \tau_x' - \rho\nu u_\Psi' = -\tau'/u' \tag{2}$$

A generalized scalar c (species, enthalpy, turbulence properties, etc.) is introduced into the flow which, although dilute, modifies ν and obeys the following conservation and flux equations, analogous to Eqs. (1) and (2):

$$\rho c_x' - q_\Psi' = 0 \tag{3}$$

$$\epsilon_q' q_x' - (\rho\nu/Pr) c_\Psi' = -q'/u' \tag{4}$$

where Pr is the appropriate ratio of diffusivities. It is convenient to nondimensionalize these equations such that u and c vary between 0 and 1 across the shear layer. The resultant dimensionless form of Eqs. (1-4) (indicated by the tilde variables) is

$$\tilde{u}_{\tilde{x}} - \tilde{\tau}_{\tilde{\Psi}} = 0 \tag{5}$$

$$\tilde{c}_{\tilde{x}} - \tilde{q}_{\tilde{\Psi}} = 0 \tag{6}$$

$$\tilde{\tau}_{\tilde{x}} - (\tilde{\nu}/Re\tilde{\epsilon}_{\tilde{\tau}})\tilde{u}_{\tilde{\Psi}} = -\tilde{\tau}/\tilde{u}\tilde{\epsilon}_{\tilde{\tau}} \tag{7}$$

$$\tilde{q}_{\tilde{x}} - (\tilde{\nu}/Pe\tilde{\epsilon}_{\tilde{q}})\tilde{c}_{\tilde{\Psi}} = -\tilde{q}/\tilde{u}\tilde{\epsilon}_{\tilde{q}} \tag{8}$$

For later reference, Pr is expressed as

$$Pr = 1 - \sigma \quad \text{where} \quad \sigma \ll 1 \tag{9}$$

As a result, and for notational convenience, $\tilde{\epsilon}_{\tilde{\tau}}/\tilde{\epsilon}_{\tilde{q}}$ is set equal to one (and the subscripts dropped) inasmuch as it is likely to be a strong function of σ. If information on the individual relaxation constants is available they may each be retained with little additional complexity, or a model for the dependence of their ratio on σ may be examined. For small values of σ and $\tilde{\epsilon}$, however, this is a higher order effect. Finally, the transport model analyzed here introduces a weak $\tilde{\nu}(\tilde{c})$ dependence, such that momentum and scalar transport depend on the generalized transported scalar

$$\tilde{\nu} = (1 - \tilde{\delta}\tilde{c}); \quad \tilde{\delta} \ll 1 \tag{10}$$

with a constant value of Pr. Other functional relationships are equally amenable to analysis, including a variable Pr for which the (second-order) dependence is $\sigma = \sigma_{\tilde{c}=0} + (\tilde{\delta}_{\tilde{\tau}} - \tilde{\delta}_{\tilde{q}})\tilde{c}$.

Method of Characteristics Solution

Substituting Eqs. (9) and (10), and introducing $\lambda = d\tilde{\Psi}/d\tilde{x}$ into Eqs. (5-8), they may be cast into the following characteristic form:

$$\lambda \tilde{u}_{\tilde{\Psi}} + \tilde{\tau}_{\tilde{\Psi}} = \frac{d\tilde{u}}{d\tilde{x}} \tag{11}$$

$$\lambda\tilde{c}_{\tilde{\Psi}}+\tilde{q}_{\tilde{\Psi}}=\frac{d\tilde{c}}{d\tilde{x}} \tag{12}$$

$$\lambda\tilde{\tau}_{\tilde{\Psi}}+\tilde{\theta}\tilde{u}_{\tilde{\Psi}}=\frac{d\tilde{\tau}}{d\tilde{x}}+\frac{\tilde{\tau}}{\tilde{u}\tilde{\epsilon}} \tag{13}$$

$$\lambda\tilde{q}_{\tilde{\Psi}}+\tilde{\Phi}\tilde{c}_{\tilde{\Psi}}=\frac{d\tilde{q}}{d\tilde{x}}+\frac{\tilde{q}}{\tilde{u}\tilde{\epsilon}} \tag{14}$$

where

$$\tilde{\theta}\equiv(1-\tilde{\delta}\tilde{c})/\tilde{\epsilon}Re \tag{15}$$

$$\tilde{\Phi}\equiv\tilde{\theta}/(1-\sigma) \tag{16}$$

As a result, it may be observed that Eqs. (11-14) have the following real characteristics:

$$\pm\tilde{\theta}^{1/2}\frac{d\tilde{u}}{d\tilde{x}}-\frac{d\tilde{\tau}}{d\tilde{x}}-\frac{\tilde{\tau}}{\tilde{u}\tilde{\epsilon}}=0;\quad \frac{d\tilde{\Psi}}{d\tilde{x}}=\pm\tilde{\theta}^{1/2} \tag{17}$$

$$\pm\tilde{\Phi}^{1/2}\frac{d\tilde{c}}{d\tilde{x}}-\frac{d\tilde{q}}{d\tilde{x}}-\frac{\tilde{q}}{\tilde{u}\tilde{\epsilon}}=0;\quad \frac{d\tilde{\Psi}}{d\tilde{x}}=\pm\tilde{\Phi}^{1/2} \tag{18}$$

Equations (17) and (18) will be manipulated to expose the dependence of solutions on $\tilde{\epsilon}$, Re, $\tilde{\delta}$, and σ. By introducing the following scaling:

$$x=\tilde{x}/\tilde{\epsilon}^{1/2},\quad \delta=\tilde{\delta}/\tilde{\epsilon}^{1/2},\quad y=\tilde{y}Re^{1/2}/\tilde{\epsilon}^{1/2},\quad \Psi=\tilde{\Psi}Re^{1/2}$$

$$\tau,q=\tilde{\tau},\tilde{q}\cdot\tilde{\epsilon}Re^{1/2},\quad u,c=\tilde{u},\tilde{c}\cdot\tilde{\epsilon}^{1/2} \tag{19}$$

Eqs. (11-18) are made invariant with respect to $\tilde{\epsilon}$ and Re, from which their effect on the solution structure is open to inspection. Utilizing this transformation, expanding for small δ and σ, and retaining first-order correction terms, Eqs. (17) and (18) may be written in dimensionless scaled variables as

$$\pm 1-\frac{\delta c}{2}\frac{du}{dx}=\frac{d\tau}{dx}+\frac{\tau}{u};\quad \frac{d\Psi}{dx}=\pm 1-\frac{\delta c}{2} \tag{20}$$

$$\pm 1+\frac{\sigma}{2}1-\frac{\delta c}{2}\frac{dc}{dx}=\frac{dq}{dx}+\frac{q}{u};\quad \frac{d\Psi}{dx}=\pm 1+\frac{\sigma}{2}1-\frac{\delta c}{2} \tag{21}$$

Solution Evaluations

In order to expose the effect of δ and σ on the solution structure, consider four separate cases associated with zero and small values of δ and σ. First, we examine the two situations in which $\sigma=0$. In this limit the structure of Eqs. (20) and (21) is similar and, with appropriate initial and boundary condition similarity, the solutions are identical. This condition is adequate to analyze the impact of δ and σ on the solutions. Therefore, with $\sigma=0$, $u=c$ and $\tau=q$, and Eqs. (20) and (21) may be evaluated with respect to δ. If $\delta=0$, Eqs. (20) and (21) degenerate to

$$\pm\frac{du}{dx}=\frac{d\tau}{dx}+\frac{\tau}{u};\quad \frac{d\Psi}{dx}=\pm 1 \tag{22}$$

This is the result from Ref. 4. Considering situations for which u is everywhere nonzero (for example in a mixing layer between two flowing streams), Eq. (22) may be integrated to provide

$$\tau\exp[\int dx/u]\pm\int\exp[\int dx/u]du=\text{const} \tag{23}$$

In the far field, Eq. (23) indicates that

$$\tau\pm u\sim e^{-x} \tag{24}$$

The effect of small δ on this result may be obtained by considering Eqs. (20) and (21), written as

$$\pm 1-\frac{\delta u}{2}\frac{du}{dx}=\frac{d\tau}{dx}+\frac{\tau}{u};\quad \frac{d\Psi}{dx}=\pm 1-\frac{\delta u}{2} \tag{25}$$

If we expand u and τ in a power series in δ: $u=u^0+\delta u^1$, $\tau=\tau^0+\delta\tau^1$, it is clear that the u^0 solution is Eq. (23). By collecting terms of order δ, the following equation for u^1 and τ^1 may be derived:

$$\pm\frac{du^1}{dx}+\frac{\tau^0u^1}{u^{0^2}}=\frac{d\tau^1}{dx}+\frac{\tau^1}{u^0}+\frac{d(u^{0^2}/4)}{dx} \tag{26}$$

which may be integrated to give

$$\tau^1\exp[\int dx/u^0]+\int\exp[\int(u^0+\tau^0)dx/u^{0^2}]$$
$$\times d\{u^1\exp[\pm\int\tau^0dx/u^{0^2}]\}$$
$$=\pm\int\exp[\int dx/u^0]d(u^{0^2}/4)+\text{const} \tag{27}$$

In the far field, Eq. (27) reduces to

$$\tau^1\pm u^1\sim e^{-2x} \tag{28}$$

Therefore, the effect of δ is to introduce a new functional dependence such that

$$\tau\pm u\sim e^{-x}+\delta e^{-2x} \tag{29}$$

along slightly distorted characteristics: $d\Psi/dx=\pm(1-\delta u/2)$.

We now consider cases for which σ is small but nonzero. Initially, let $\delta=0$ for which Eqs. (20) and (21) become

$$\pm\frac{du}{dx}=\frac{d\tau}{dx}+\frac{\tau}{u};\quad \frac{d\Psi}{dx}=\pm 1 \tag{30}$$

$$\pm 1+\frac{\sigma}{2}\frac{dc}{dx}=\frac{dq}{dx}+\frac{q}{u};\quad \frac{d\Psi}{dx}=\pm 1+\frac{\sigma}{2} \tag{31}$$

The solution to Eq. (30) is, once again, contained in Eq. (23). In addition, since u is here independent of c, Eq. (31) obeys this same functional form with $(1+\sigma/2)c$ and q replacing u and τ, respectively. Therefore, although the functional form for the c decay is retained, the numerical constant in front of the exponential is smaller. Of prime importance, however, is that this decay occurs along steeper characteristics. Therefore, the effect of (u and c) data with nonzero σ is dispersed throughout the flow *by the c equation* more rapidly than with zero σ.

Finally, the case of nonzero σ and δ is examined. By expanding Eqs. (20) and (21) first in a series in σ (e.g., $u=u^0+\sigma u^1$), it is readily apparent that u^0 satisfies Eq. (25). By collecting terms of order σ, the following equations are derived:

$$\pm 1-\frac{\delta c^0}{2}\frac{du^1}{dx}+\frac{\tau^0u^1}{u^{0^2}}=\frac{d\tau^1}{dx}+\frac{\tau^1}{u^0} \tag{32}$$

$$\pm 1-\frac{\delta c^0}{2}\frac{dc^*}{dx}+\frac{q^0u^1}{u^{0^2}}=\frac{dq^1}{dx}+\frac{q^1}{u^0},\quad c^*\equiv c^1+\frac{c^0}{2} \tag{33}$$

Inasmuch as (u^l, c^l) is a σ correction and $\delta, \sigma \ll 1$, (u^l, c^l) may be expanded in a series in δ and evaluated at order δ^0 to obtain a valid first-order correction. The resultant equations may be integrated to provide

$$\tau^l \exp[\int dx/u^0] + \int \exp[\int (u^0 + \tau^0) dx/u^{0^2}] \times d\{u^l \exp[\pm \int \tau^0 dx/u^{0^2}]\} = \text{const} \quad (34)$$

$$q^l \exp[\int dx/u^0] + \int \exp[\int dx/u^0] d(c^l + c^0/2) = \int q^0 u^l \exp[\int dx/u^0] dx/u^{0^2} + \text{const} \quad (35)$$

which, in the far field, provides an additional exponential dependence proportional to σ.

Of primary importance here is the interpretation that the effect of δ, in real flows for which $\sigma > 0$, is to allow the transported scalar to communicate velocity field details throughout the field more rapidly than the momentum equation. In particular, due to the fact that the u equation is c dependent, information on localized peculiarities in u field data is transmitted to, and affects the solution of, the (remote) u field solution via the transported scalar.

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Noniterative Cross-Flow Integration for the Pressure-Split Analysis of Subsonic Mixing-Layer Problems

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Introduction

THE spatial marching analysis of subsonic, quasiparabolic mixing-layer problems is commonly performed using the numerical artifice of pressure splitting.[1-3] In the pressure-split approach, the governing parabolized Navier-Stokes (PNS) equations are spatially integrated with the streamwise pressure gradient "imposed," and the cross-flow pressure variation determined aposteriori, at each integration step, from the coupled solution of the continuity and cross-flow momentum equations. Thus, the stepwise integration is comprised of 1) a standard parabolic integration yielding the streamwise component of velocity and pertinent scalar (total enthalpy, species) and turbulence model variables; and 2) an elliptic-like cross-flow integration yielding the cross-flow velocity components and pressure variation. Subsequent upgrades to the solution based on a global pressure iteration can be performed in regions with strong pressure gradients.

While the parabolic streamwise integration procedures are comparable in most models, the details of the local and global pressure-splitting procedures are problem dependent, and the cross-flow solution techniques vary widely. The continuity and cross-flow momentum equations are strongly coupled through the pressure, density, and cross-flow velocity derivatives, and are solved in an iterative manner. In the popular approach of Patankar and Spalding,[1] a pressure-correction equation arrived at from the continuity equation (with cross-flow momentum constraints) is used to determine the cross-flow pressure variation, while the cross-flow velocities are determined from the momentum equations. In contrast, the two-dimensional iterative procedure of Bradshaw and coworkers[3,4] employs the continuity equation to determine the cross-flow velocity and the normal momentum equation to determine the pressure variation.

In utilizing pressure-split methodology for the two-dimensional analysis of curved wall jets[5] and subsonic regions of underexpanded free jets[6,7] (i.e., behind Mach disks and between the jet mixing-layer sonic line and jet outer edge), Dash and coworkers initially utilized the cross-flow procedure of Bradshaw and coworkers. In assessing this procedure, it was found that the iterative sweeps required between the continuity and cross-flow momentum equation solutions could be eliminated by combining these equations into a unified equation for the cross-flow velocity. This provides a considerable savings in overall computer time and eliminates possible convergence problems occurring in iterative approaches. This Note describes this new noniterative procedure and its application to a simple two-dimensional curved wall jet problem.

It should be noted that this new technique extends the efficiency of single-sweep pressure-splitting methodology to a level whereby PNS mixing solutions can be obtained in slightly more time than that required for standard parabolic mixing solutions. The results provided using this rapid procedure are identical to those obtained using previous iterative, pressure-split methodology as ascertained by numerical experiments. Hence, the favorable comparisons with data obtained using the curved wall jet and free jet models of Refs. 5-7 remain unchanged, and, the results of Bradshaw and coworkers[3,4] would be reproduced using this approach if all other aspects of the computational procedure and turbulence modeling were duplicated.

Cross-Flow Analysis

The overall mixing-layer analysis is performed in mapped, surface-oriented curvilinear coordinates using the upwind, implicit formulation described in Ref. 5. With the rectangular mapping

$$\xi = s \qquad \eta = n/\delta(s) \quad (1)$$

(where s is the streamwise direction, n is normal to it, and $\delta(s)$ is the width of the mixing zone) the parabolized, planar normal momentum equation can be written[5]:

$$\boxed{\rho U \frac{\partial V}{\partial \xi} + \rho \tilde{V} \frac{\partial V}{\partial \eta} + bh \frac{\partial P}{\partial \eta} + K\rho U^2 = g_v} \quad (2)$$

where $\tilde{V} = bhV - aU$; K is the curvature, h a curvature parameter $(= 1 - nK)$; a and b mapping parameters; and g_v contains the laminar and turbulent stress terms.[5]

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